Math 279 Lecture 16 Notes

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1 Proof of Hairer's Reconstruction Theorem

1.1 Motivation for multiresolution analysis

We wish to show the following reconstruction theorem of Martin Hairer:

Theorem 1.1. *If F is a γ-coherent germ, then there exists a distribution T such that*

$$|\langle T - F_x, \varphi_x^\delta \rangle| \lesssim \delta^\gamma, \qquad \gamma \neq 0.$$

This is uniform for $\delta \in (0,1]$, $x \in K$, $\operatorname{supp} \varphi \subseteq B_1(0)$, $\|\varphi\|_{C^r} \leq 1$.

Last time, we showed that T is unique if $\gamma > 0$. However, if $\gamma < 0$, then we can add a distribution S to T, provided that

$$|\langle S, \varphi_x^\delta \rangle| \lesssim \delta^\gamma,$$

which means $S \in \mathcal{C}^\gamma$.

To give an idea about the strategy of the proof, we first discuss Hairer's original proof that uses wavelet expansion. In fact, the proof we presented for $d = 1$ uses the wavelet $\mathbb{1}_{[0,1]}$, i.e. the Haar basis. Recall that if $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$, then

$$\begin{aligned}
\int_s^t f g' \, d\theta &= \underbrace{(f g')}_{T}(\mathbb{1}_{[s,t]}) \\
&\approx \sum_{t_i^n \in [s,t]} f(t_i^n)(g(t_{i+1}^n) - g(t_i^n)) \\
&= \sum_{t_i^n \in [s,t]} f(t_i^n) g'(\mathbb{1}_{[t_i^n, t_{i+1}^n]}) \\
&= \sum_{t_i^n \in [s,t]} F_{t_i^n}(\mathbb{1}_{[t_i^n, t_{i+1}^n]}).
\end{aligned}$$

And we have shown that this converges if $\alpha + \beta > 1$. For our extension, we replace $\mathbb{1}_{[s,t]}$ with $\varphi \in \mathcal{D}$, and the Haar basis may be replaced with a basis using a multiresolution analysis (MRA) of Mallat.

1.2 Multiresolution analysis

Here is a quick review of MRA:

Definition 1.1. We say $\phi \in L^2(\mathbb{R})$ is a **scaling function** or a **(father) wavelet**[1] if the following conditions are true: First, let $\phi_a^n(x) = 2^{n/2}\phi(2^n(x-a))$, where $n \in \mathbb{Z}$, $a \in 2^{-n}\mathbb{Z}$ so that $\|\phi_a^n\|_{L^2} = \|\phi\|_{L^2}$. Also set $V_n = \operatorname{span}\{\phi_a^n : a \in \Lambda_n = 2^{-n}\mathbb{Z}\}$.

(i) $V_n \subseteq V_{n+1}$ (it suffices to have $V_0 \subseteq V_1$)

(ii) $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0 (hence $\{\phi_a^n : a \in \Lambda_n\}$ is an orthonormal basis for V_n)

(iii) $L^2(\mathbb{R}) = \overline{\bigcup_n V_n}$.

Example 1.1. We can take, for example, $\phi = \mathbb{1}_{[0,1]}$ to get functions of the form $\phi_a^n = \mathbb{1}_{[t_i^n, t_{i+1}^n]}$. Also, $V_0 = \{\phi(\cdot - k) : k \in \mathbb{Z}\}$.

Remark 1.1. It can be proved that there is no such ϕ which is smooth and has compact support. However, if we only require that ϕ has a certain number of derivatives, it is possible to construct one.

Remark 1.2. We may find W_n such that $V_{n+1} = V_n \oplus W_n$ (W_n is the orthogonal complement of V_n inside V_{n+1}).

Proposition 1.1. *There exists ψ such that if $\psi_a^n(x) = 2^{n/2}\psi(2^n(x-a))$, then*

$$W_n = \operatorname{span}\{\psi_a^n : a \in \Lambda_n\}.$$

*This ψ is called the **(mother) wavelet**.*

Remark 1.3. In fact, it suffices to find $\psi \in V_1$ so that ψ is orthogonal to the integer translates of ϕ, and $W_0 = \operatorname{span}\{\psi(\cdot - k) : k \in \mathbb{Z}\}$. Indeed,

$$V_0 \subseteq V_1 \iff \phi(x) = \sqrt{2}\sum_{r\in\mathbb{Z}} a_r \phi(2x - r) \text{ for coefficients } a_r,$$

And ψ is simply given by

$$\psi(x) = \sqrt{2}\sum_{r\in\mathbb{Z}} b_r \phi(2x-r), \qquad b_r = (-1)^r a_{1-r}.$$

Example 1.2. When $\phi = \mathbb{1}_{[0,1]}$, we may take to be 1 on $[0, 1/2]$ and -1 on $[-1/2, 0)$.

Here is the proof of $\psi \perp V_0$:

[1] There are also mother wavelets.

Proof. Observe that

$$\begin{aligned}\phi_\ell(x) &= \phi(x-\ell)\\ &= \sum_r a_r(\sqrt{2}\phi(2x-2\ell-r))\\ &= \sqrt{2}\sum_r a_{r-2\ell}\phi(2x-r).\end{aligned}$$

Hence,

$$\begin{aligned}\langle\psi,\phi_\ell\rangle &= \sum_r a_{r-2\ell}b_r\\ &= \sum_r a_{r-2\ell}(-1)^r a_{1-r}\end{aligned}$$

Denote $1-s=r-2\ell$

$$= -\sum_s a_{1-s}(-1)^s a_{s-2\ell},$$

which implies that $\langle\psi,\phi_\ell\rangle = 0$. □

Theorem 1.2 (Ingrid Daubechies)**.** *For every k, there exists a scaling function $\phi \in C^k$ of compact support. Moreover, any polynomial of degree k is in V_0.*

1.3 Strategy of Hairer's proof of the reconstruction theorem

Assuming this theorem of Daubechies, we are now ready to describe Hairer's strategy for the proof. Again, we wish to find a distribution T such that $\langle T-F_x,\varphi_x^\delta\rangle \lesssim \delta^\gamma$. Here, is the recipe for constructing T: When $\gamma>0$, $T=\lim_{n\to\infty}T_n$ (this means for every $\psi\in\mathcal{D}$, $T(\psi)=\lim_{n\to\infty}T_n(\psi)=\lim_{n\to\infty}\int T_n(x)\psi(x)\,dx$), where

$$T_n(x)=\sum_{a\in\Lambda_n}\langle F_a,\phi_a^n\rangle\phi_a^n(x).$$

How about $\gamma<0$? In this case, the convergence fails. Recall that if $n>0$,

$$V_n = V_{n-1}\oplus W_{n-1} = V_{n-2}\oplus W_{n-1}\oplus W_{n-2} = \cdots = V_0\oplus W_0\oplus W_1\oplus\cdots\oplus W_{n-1}.$$

Hence, $L^2=\overline{V_0\oplus\bigoplus_{n=0}^\infty W_n}$, or more generally,

$$L^2=\overline{V_m\oplus\bigoplus_{n=m}^\infty W_m}.$$

So for any u,

$$u=\sum_{a\in\Lambda_m}\langle u,\phi_a^m\rangle\phi_a^m+\sum_{n=m}^\infty\sum_{a\in\Lambda_m}\langle u,\psi_a^n\rangle\psi_a^n.$$

Our candidate for T is

$$T = \sum_{a \in \Lambda_m} \langle F_a, \phi_a^m \rangle \phi_a^m + \sum_{n=m}^{\infty} \sum_{a \in \Lambda_m} \langle F_a, \psi_a^n \rangle \psi_a^n.$$

1.4 Proof of the reconstruction theorem without wavelet expansions

We now present a proof that does not use wavelet expansions. We achieve this by using a suitable $\rho \in \mathcal{D}$. If we choose ρ correctly, then

$$T_n = F_x(\widehat{\rho}_x^n), \qquad \text{where } \widehat{\rho}_x^n(y) = 2^{dn}\rho(2^n(x-y)) = \rho_x^{2^{-n}}(y).$$

For $\gamma > 0$, the limit $\lim_n T_n$ will exist, but for $\gamma < 0$, we will throw away a "bad term" which will not matter. We will finish the explanation next time.